

Feynman rules for Coulomb gauge QCD

A. Andraši ^a, J. C. Taylor ^b

March 2, 2013

a Rudjer Bošković Institute, Zagreb, Croatia

b Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, UK

Keywords: Coulomb gauge, QCD, renormalization

Abstract

The Coulomb gauge in nonabelian gauge theories is attractive in principle, but beset with technical difficulties in perturbation theory. In addition to ordinary Feynman integrals, there are, at 2-loop order, Christ-Lee (CL) terms, derived either by correctly ordering the operators in the Hamiltonian, or by resolving ambiguous Feynman integrals. Renormalization theory depends on the sub-graph structure of ordinary Feynman graphs. The CL terms do not have a sub-graph structure. We show how to carry out renormalization in the presence of CL terms, by re-expressing these as ‘pseudo-Feynman’ integrals. We also explain how energy divergences cancel.

1 Introduction

There are some reasons to be interested in the Coulomb gauge in QCD. It is the only explicitly unitary gauge (we discount axial gauges, which have undefined denominators $1/(n.k)^1$). It may be useful for bound-state problems. It has been used in lattice simulations, and it has been made the basis for a discussion of confinement [1], [2], [3], [4], [5], [6], [7], [8], [13], [9], [10],[11] (the existence of a Hamiltonian allows the use of a variational principle [12]). But there are complications in formulating correct Feynman rules, all concerned with the convergence of integrals over the energy-components of internal energy-momentum variables. These problems are eased by using the Hamiltonian (phase-space) formulation, in which the electric field \mathbf{E}^a is one of the dynamical variables [14]. The Lagrangian in the Hamiltonian formalism, and the Feynman rules are stated in Appendix A.

¹We include here the temporal gauge, in which n is time-like, although this gauge is sometimes taken as a starting point.

Even with this formulation, divergent energy integrals remain in individual Feynman graphs, which are cancelled when suitable sets of graphs are combined. This has been illustrated in [16].

A more subtle complication about the energy integrals remains. This was first recognized as a question of operator ordering by [15] (which contains references to earlier related work, and see also [18]). But an equivalent approach is to regard it as coming from ambiguous double energy integrals at 2-loop order (see [19]). This integral is (we use bold capital letters for the space components of 4-vectors)

$$\int dp_0 dq_0 \frac{p_0}{p_0^2 - \mathbf{P}^2 + i\epsilon} \frac{q_0}{q_0^2 - \mathbf{Q}^2 + i\epsilon} \quad (1.1)$$

This double integral is not absolutely convergent, and might be defined as a repeated integral in two different ways:

$$\int dp_0 \frac{p_0}{p_0^2 - \mathbf{P}^2 + i\epsilon} \int dq_0 \frac{q_0}{q_0^2 - \mathbf{Q}^2 + i\epsilon} = 0 \quad (1.2)$$

or

$$\int \frac{dr_0}{2\pi} \int \frac{dq_0}{2\pi} \frac{-i(r_0 + q_0)}{(r_0 + q_0)^2 - \mathbf{P}^2 + i\epsilon} \frac{iq_0}{q_0^2 - \mathbf{Q}^2 + i\epsilon} = 1/4. \quad (1.3)$$

The above ambiguous integrals occur in 2-loop order in graphs containing just two propagators $\langle 0 | T(A_i^a(x) E_j^b(y)) | 0 \rangle$ (our notation is explained in Appendix A) and a connected chain of Coulomb propagators.

Christ and Lee [15] (and previous authors referenced there) studied the correct ordering of operators (including the Fadeev-Popov determinant) in the Hamiltonian. They found two new terms called V_1 and V_2 (of order \hbar^2) which had to be added. At the same time, the ordinary parts of the Hamiltonian were to be Weyl ordered (that is a certain average over different orderings). It was argued that this Weyl ordering removed all the ambiguous integrals like (1.1), their place being taken by V_1 and V_2 .²

An alternative approach was given in [20]. There, the ambiguity of (1.1) was resolved by showing that suitable combinations of Feynman graphs lead to the unambiguous, absolutely convergent, combination (defining $p_0 + q_0 + r_0 = 0$)

$$\begin{aligned} \int \frac{dp_0}{2\pi} \frac{dq_0}{2\pi} \left(\frac{ip_0}{p_0^2 - \mathbf{P}^2 + i\epsilon} \frac{iq_0}{q_0^2 - \mathbf{Q}^2 + i\epsilon} + \frac{iq_0}{q_0^2 - \mathbf{Q}^2 + i\epsilon} \frac{ir_0}{r_0^2 - \mathbf{R}^2 + i\epsilon} \right. \\ \left. + \frac{ir_0}{r_0^2 - \mathbf{R}^2 + i\epsilon} \frac{ip_0}{p_0^2 - \mathbf{P}^2 + i\epsilon} \right) = 1/4. \end{aligned} \quad (1.4)$$

This double integral may be done in any order, and therefore confirmed by using (1.2) and (1.3).

Equation (1.4) is related to the identity

$$\epsilon(t_1 - t_2)\epsilon(t_2 - t_3) + \epsilon(t_2 - t_3)\epsilon(t_3 - t_1) + \epsilon(t_3 - t_1)\epsilon(t_1 - t_2) = -1,$$

²In [15], (section VI) it is stated that the integral (1.7) is zero. There seems to be no explicit statement about integrals like (1.1). If the Weyl ordering implies that all integrals like (1.1) are zero, there seems to lead to a contradiction because of the identity (1.4).

which has an unambiguous limit as the times tend to zero.

Actually, (1.4) is not quite sufficient to resolve all ambiguity. In [20] one further rule was used. This is that integrals of the form

$$X \equiv \int dq_0 d\mathbf{Q} dr_0 d\mathbf{R} \frac{q_0}{q_0^2 - \mathbf{Q}^2 + i\epsilon} \frac{r_0}{r_0^2 - \mathbf{R}^2 + i\epsilon} f(\mathbf{Q}) g(\mathbf{R}) = 0 \quad (1.5)$$

where f, g are any two functions. This rule could not be obtained from within the Coulomb gauge, but was deduced from the limit of a gauge which interpolates between the Feynman gauge and the Coulomb gauge. The argument is that, in the interpolating gauge, the double integrand still factorizes, taking the form

$$\int dq_0 d^3\mathbf{Q} dr_0 d^3\mathbf{R} F(\mathbf{Q}, q_0^2) G(\mathbf{R}, r_0^2), \quad (1.6)$$

where each energy integral is now convergent, and so is unambiguously zero.³

From the identity (1.4) and the rule (1.5), it was shown (provided that the external gluons are transverse) in [20] that the additions $V_1 + V_2$ of [15] are contained within ordinary Feynman integrals.

The problem with the above treatments of the ambiguous integrals is as follows. It is an important property of Feynman graphs that, as a simple example, a 1-loop graph may appear as a sub-graph of a 2-loop graph; and when it does it represents the same integral in both cases. This property is vital to the process of renormalization of ultra-violet (UV) divergences. The necessary counter-terms are found from 1-loop integrals, and are then used to cancel the sub-divergences at 2-loops.

In the presence of the ambiguous integrals (1.1), this property is not obvious. For example, the 1-loop graph in Fig.5(c) contains the integral

$$\int dp_0 \frac{p_0}{p_0^2 - \mathbf{P}^2 + i\epsilon} \quad (1.7)$$

which is naturally, and correctly, taken to be zero. But in the 2-loop graph Fig.6(a), which contains Fig.5(c) as a sub-graph, the ambiguous integral (1.1) appears, which according to [20] has to be combined with other graphs to give (1.4). It is not obviously correct to attach the value zero to Fig.5(c) in isolation. What is more, if the sub-graph in Fig.6(a) is not zero, it is UV divergent, but the calculation of Fig.5(c) provides no counter-term to cancel any sub-divergence in Fig.6(a). In a similar way, the 1-loop graph in Fig.5(a) is unambiguous (and UV divergent), but it appears as a sub-graph in Fig.6(b) which contains the ambiguous double integral (1.1).

This is a rather trivial example, but in the following we will encounter several similar cases.

It is the purpose of this paper to propose a solution to the dilemma.

We first have to remind the reader what the functionals V_1 and V_2 in [15] are.

³There is no actual Feynman graph giving an integral like (1.5). There are (non-zero) Feynman graphs which give integrals like (1.5) but with r_0 replaced by p_0 (as in Fig.8(c)), and (1.5) is just something which is added on to these in order to get the combination in (1.4).

2 The functionals of Christ and Lee

In order to state what these are, we first need to define the ghost propagator, G . It is defined to be the solution of the equation

$$(\partial/\partial X_i)(D_i^{ab}(\mathbf{X})G^{bc}(\mathbf{X}, \mathbf{X}'; \mathbf{A})) = \delta^{ac}\delta^3(\mathbf{X} - \mathbf{X}'). \quad (2.1)$$

Here i, j, \dots are 3-vector indices, a, b, \dots are colour indices, we write a 4-vector $x_\mu = (t, X_i)$, and D_i^{ab} is defined in (A3). In (2.1), $A_i^a(x)$ is an external gluon field which is transverse, $\partial A_i^a(x)/\partial X_i = 0$. G is a functional of A_i^a . Note that (2.1) is an instantaneous equation, there are no time derivatives. It is convenient to define also

$$G_i^{ab}(\mathbf{X}, \mathbf{X}'; \mathbf{A}) \equiv (\partial/\partial X_i)G^{ab}(\mathbf{X}, \mathbf{X}'; \mathbf{A}). \quad (2.2)$$

The zeroth order term in the perturbation expansion of G is just the Coulomb potential

$$G^{ab} = \delta^{ab} \frac{1}{4\pi} \frac{1}{|\mathbf{X} - \mathbf{X}'|} + O(g) \equiv \delta^{ab} G_0(\mathbf{X} - \mathbf{X}') + O(g). \quad (2.3)$$

In terms of the ghost propagator, the complete Coulomb potential operator is

$$C^{ab}(\mathbf{X}, \mathbf{X}'; \mathbf{A}) = \int d^3\mathbf{X}'' G_i^{ca}(\mathbf{X}'', \mathbf{X}; \mathbf{A}) G_i^{cb}(\mathbf{X}'', \mathbf{X}'; \mathbf{A}). \quad (2.4)$$

We need also to define the transverse projection operator

$$T_{ij}(\mathbf{X}, \mathbf{Y}) = \delta_{ij}\delta^3(\mathbf{X} - \mathbf{Y}) - \partial_i\partial_j G_0(\mathbf{X} - \mathbf{Y}). \quad (2.5)$$

With this notation, the first Christ-Lee contribution to the Hamiltonian is

$$V_1 = -(g^2/8) f^{abc} f^{a'b'c'} \int d^3\mathbf{X} d^3\mathbf{Y} \\ \times G_i^{aa'}(\mathbf{X}, \mathbf{Y}; \mathbf{A}) \delta^3(\mathbf{X} - \mathbf{Y}) \delta^{bb'} G_i^{c'e}(\mathbf{Y}, \mathbf{X}; \mathbf{A}) \quad (2.6)$$

For the second, we first define

$$L_{ij}^{ab}(\mathbf{X}, \mathbf{Y}; \mathbf{A}) \equiv \delta^{ab} T^{ij}(\mathbf{X}, \mathbf{Y}) + g \int d^3\mathbf{Z} G_i^{ac}(\mathbf{X}, \mathbf{Z}; \mathbf{A}) f^{cbe} A_k^e(\mathbf{Z}) T_{kj}(\mathbf{Z}, \mathbf{Y}) \quad (2.7)$$

Then

$$V_2 = -(g^2/8) f^{abc} f^{a'b'c'} \int d\mathbf{X} d\mathbf{Y} L_{ij}^{aa'}(\mathbf{X}, \mathbf{Y}; \mathbf{A}) C^{bb'}(\mathbf{X}, \mathbf{Y}; \mathbf{A}) L_{ji}^{c'e}(\mathbf{Y}, \mathbf{X}; \mathbf{A}). \quad (2.8)$$

Here we have not used the original form in [15] but an equivalent given by [20] in his equation (4.5.3).

The contributions to (2.6) and (2.8) in a perturbation series may be represented by 3-dimensonal Feynman graphs, in terms of the Fourier transform

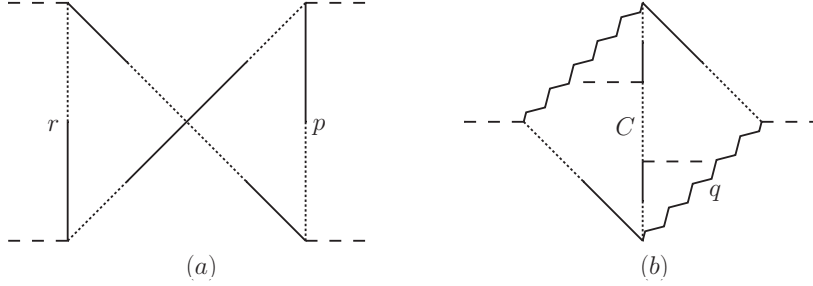


Figure 1: Three-dimensional graphs of order g^6 in momentum space. (a) an example of a contribution to V_1 in (2.6). (b) an example of a contribution to V_2 in (2.8). The central chain of lines is from C in (2.4) and we call this the C -chain. The two side chains are each from L . The zig-zag line denotes (the Fourier transform of) T_{ij} . The symbol C on the central segment in (b) draws attention to the Coulomb propagator G_0 in (2.3), which in this case could be in any one of three positions on the central vertical C -chain, giving three equal contributions.

$A_i^a(\mathbf{K})$ of $A_i^a(\mathbf{X})$ (which we assume to satisfy $\mathbf{K} \cdot \mathbf{A}^a = 0$ ⁴). We shall call these CL graphs. Typical examples, to order g^6 are shown in Fig.1(a) for V_1 and Fig.1(b) for V_2 . Our graphical notation is explained in Appendix A. The rules are like those for a Euclidean field theory in 3-dimensions.

It will be useful to have a notation for the CL terms in momentum space, like the examples in Fig.1. We will denote these integrals by $V_1(N)$ and $V_2(N)$ where N is the total number of external gluon lines. Then

$$\begin{aligned}
 -V_1(N) &= \frac{g^{N+2}}{(2\pi)^6} f^{abc} f^{a'bc'} \int d^3\mathbf{P} d^3\mathbf{R} \sum_{n,n'} \tilde{G}_i^{aa'}(\mathbf{P}, \{n\}) \tilde{G}_i^{c'c}(\mathbf{R}, \{n'\}) \\
 &\equiv \frac{1}{(2\pi)^6} \int d^3\mathbf{P} d^3\mathbf{R} W_1^N(\mathbf{P}, \mathbf{R}),
 \end{aligned} \tag{2.9}$$

where n stands for the quantum numbers of n external gluons, that is for $\{\mathbf{K}_\nu, i_\nu, a_\nu\}$ with $\nu = 1, \dots, n$ and $n + n' = N$. Thus Fig.1(a) is an example with $n = n' = 2$. We have put the minus sign on the left in (2.9) because $-V_{1,2}$ are the contributions to the effective action.)

⁴This condition is assumed also in the work of Christ and Lee. Thus, unlike the contributions form ordinary Feynman diagrams, the CL part of the effective action is not known for non-transverse external gluon fields.

Similarly,

$$\begin{aligned}
-V_2(N) &= \frac{g^{N+2}}{(2\pi)^6} f^{abc} f^{a'b'c'} \\
&\times \int d^3\mathbf{P} d^3\mathbf{Q} d^3\mathbf{R} \delta^3(\mathbf{P}+\mathbf{Q}+\mathbf{R}) \sum_{n,n',n''} (n''+1) \tilde{L}_{ij}^{aa'}(\mathbf{P}, \{n\}) \tilde{C}^{bb'}(\mathbf{R}, \{n''\}) \tilde{L}_{ji}^{cc'}(\mathbf{Q}, \{n'\}) \\
&\equiv \frac{1}{(2\pi)^6} \int d^3\mathbf{P} d^3\mathbf{Q} W_2(\mathbf{P}, \mathbf{Q}, \mathbf{R}), \tag{2.10}
\end{aligned}$$

where here $n + n' + n'' = N$. Fig.1(b) is an example with $n = n' = 1, n'' = 2$. The factor $(n'' + 1)$ is inserted into (2.10) to take account of the $n'' + 1$ possible positions of the zero-order Coulomb propagator (2.3) within \tilde{C} (as indicated by the ‘C’ in Fig.1(b).) In (2.10), \mathbf{P} is defined to be the momentum through the transverse projector $\{\delta_{ij} - (P_i P_j / \mathbf{P}^2)\}$ in $L(\mathbf{P})$, and \mathbf{Q} likewise. \mathbf{R} is defined so that $\mathbf{P} + \mathbf{Q} + \mathbf{R} = 0$.

The theorem proved in [20] may be stated as follows: the sum of the ambiguous Feynman integrals (containing (1.1)) may be brought to the form (after some manipulation)

$$\begin{aligned}
&\left[\int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \left(\frac{p_0}{p_0^2 - \mathbf{P}^2 + i\epsilon} \frac{q_0}{q_0^2 - \mathbf{Q}^2 + i\epsilon} + \frac{r_0}{r_0^2 - \mathbf{R}^2 + i\epsilon} \frac{p_0}{p_0^2 - \mathbf{P}^2 + i\epsilon} \right. \right. \\
&\quad \left. \left. + \frac{r_0}{r_0^2 - \mathbf{R}^2 + i\epsilon} \frac{q_0}{q_0^2 - \mathbf{Q}^2 + i\epsilon} \right) (W_1^N + W_2^N) \right] + X \tag{2.11}
\end{aligned}$$

where W_1^N and W_2^N are defined in (2.9) and (2.10) and X is an integral of the form of (1.5), and which is taken to be zero. The integrals in (2.11) are 4-dimensional; but not all the terms are Feynman integrals generated from the rules in the appendix, because the denominator in (1.7) can appear without the transverse projector $\{\delta_{ij} - (P_i P_j / \mathbf{P}^2)\}$ in the numerator. Also, since (1.4) is independent of $\mathbf{P}, \mathbf{Q}, \mathbf{R}$, the identification of these with the integration variables in (2.9) and (2.10) is arbitrary.

If dimensional regularization is used, with $3 - \epsilon$ spatial dimensions, the 1-loop sub-integrations in CL graphs like those in Fig.1 have no divergences, that is no poles at $\epsilon = 0$. The complete 2-loop integrals are UV divergent, with single poles at $\epsilon = 0$. At first sight, it is unexpected that there are no divergent sub-graphs in CL graphs. But these sub-graphs have no meaning on there own; there are no 1-loop CL graphs.

As explained in section 1, the problem is how to combine the CL graphs with ordinary Feynman graphs, particularly for the purpose of renormalization. In the next section, we propose an answer to this dilemma. ⁵

⁵This problem is briefly alluded to in [15] at te end of section VI. It is noted that calculation of an order g^4 contribution to V_1 involves singular terms of the form $\delta^3(\mathbf{X})/|\mathbf{X}|$. It is suggested that these may be ‘relevant to the cancellation of divergences from the usual two-loop Feynman graphs’. Using dimensional regularization, we do not encounter such singular terms.

3 Expressing Christ-Lee terms as Feynman-like integrals

We want to make (2.11) more like ordinary Feynman integrals. For the W_2^N term this is easy, but for the W_1^N term it can only be achieved partially.

The order of the p_0, q_0 integrations in (2.11) is optional, because of the identity (1.4). Let us choose the order to be that in (1.3). Then only the first term in the square bracket (2.11) contributes; the other two terms are zero because of equations like (1.2). Then, in the contribution from W_2^N , there appears the correct Feynman propagator

$$\left(\delta_{ij} - \frac{P_i P_j}{\mathbf{P}^2} \right) \frac{p_0}{p_0^2 - \mathbf{P}^2 + i\epsilon}, \quad (3.1)$$

and similarly for \mathbf{Q} . An example is shown in Fig.2(a). This looks like an ordinary Feynman graph, but it is supplemented with the rule that the ambiguous integral (1.1) is in this case to be interpreted in the order in (1.3). Thus we are to some extent reversing the process carried out in [20]. But now we attribute the value of (1.1) to a single type of graph, rather than to the combination in (2.11). The instantaneous propagators (denoted by dotted lines) are the same in the graphs in Fig.2 as in the CL graph in Fig.1(b). We will call graphs like Fig.2(a) 'pseudo-Feynman graphs'.

The above process does not generate any graph like the one in Fig.2(b), although this looks like an ambiguous Feynman graph. Thus we may say that pseudo-Feynman graphs like Fig.2(b) are zero,

Note also that the above rule is consistent with the term X in (2.11) being zero (because of (1.2)).

The general rule is that the energy integrals are to be done in the order with the energy on the C -chain of lines (defined to be a factor like C as defined in (2.4)) last. The C -chain is the central one in Fig.2(a) but the right hand chain in Fig.2(b) (it is the chain which contains a segment with just the Coulomb propagator $1/\mathbf{K}^2$). However, there are some special exceptions to this second rule, arising from V_1 .

We have not been able to express all of the W_1^N in (2.11) in terms of pseudo-Feynman graphs, but just some special parts of W_1^N . But these will turn out to be sufficient to treat renormalization. Fig.3(a) shows an example of a class of diagrams of the form of the CL graphs in Fig.1(a), which have any number of external gluon lines on the left, but only one on the right. The CL integral has the form

$$\frac{g^{3+n''}}{4} \int \frac{d^3 \mathbf{R}}{(2\pi)^3} F_k(\mathbf{R}, \mathbf{K}_j) \int \frac{d^3 \mathbf{Q}}{(2\pi)^3} \frac{Q_k}{\mathbf{Q}^2} \frac{Q'_i}{\mathbf{Q}'^2} \quad (3.2)$$

where the details of the function F are not important (and n'' is the number of external gluons in F). Because the external gluon on the right is transverse, we can replace Q'_i by $-Q_i$. We can also insert a term $\delta_{ik} \int d^3 \mathbf{Q} (1/\mathbf{Q}'^2)$ which

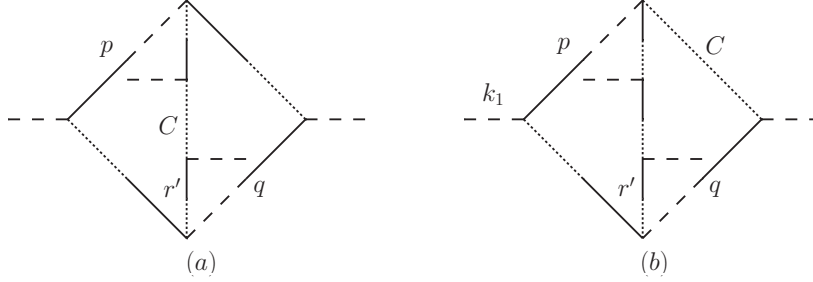


Figure 2: (a) is a pseudo-Feynman graph which is equivalent to the CL graph in Fig.1(b), provided that the energy integrals in the inserted factors are done in the order in (1.3). (b) is a Feynman graph of a similar type, but the C -chain (which in this case reduces to just a single line) is now the one on the top right, and the energy integrals are to be done in the order of (1.2), consequently giving a zero contribution. In this graph, $r' = r + k_1$ so that $p + q + r = 0$. The symbol C draws attention to the propagator G_0 in (2.3). In (a), this segment can be in any of three places, giving three equal contributions.

is zero with dimensional regularization. Thus we get instead of (3.2)

$$(1/4) \int \frac{d^3 \mathbf{R}}{(2\pi)^3} F_l(\mathbf{R}, \mathbf{K}_l) \delta_{lk} \int \frac{d^3 \mathbf{Q}}{(2\pi)^3} \left(\delta_{ik} - \frac{Q_i Q_k}{\mathbf{Q}^2} \right) \frac{1}{\mathbf{Q}^2}. \quad (3.3)$$

Now we insert the integrand in (3.3) into (2.11), and obtain

$$(2\pi)^{-8} \int d^4 r d^4 p F_l(\mathbf{R}, \mathbf{K}_l) \frac{\delta_{lk} p_0}{p_0^2 - \mathbf{P}^2} \left(\delta_{ik} - \frac{Q_i Q_k}{\mathbf{Q}^2} \right) \frac{q_0}{q_0^2 - \mathbf{Q}^2} \frac{1}{Q'^2} \quad (3.4)$$

where the p_0 and r_0 integrals are to be done in that order. (3.4) is almost a Feynman integral, except that the p -propagator has only the numerator δ_{lk} not the transverse projection operator in (3.1). This incomplete Feynman integral is denoted by the graph in Fig.3(b), where the cross bar on the p -line denotes its incomplete numerator. Apart from this feature, Fig.3(b) is a particular case of the class in Fig.2(b). Thus there are special exceptions to the statement that the terms in Fig.2(b) are zero.

(Colour factors are omitted in (3.3) and (3.4).)

4 Renormalization

Renormalization counter-terms are derived from 1-loop integrals. These have to cancel divergences in sub-graphs at 2-loops. The problem is how to identify these sub-graphs when there are CL terms which do not have the sub-graph structure of normal Feynman graphs. We will show that, re-expressing the CL integrals as pseudo-Feynman graphs, like those in Figs.2(a) and 3(b) enable us to solve this problem.

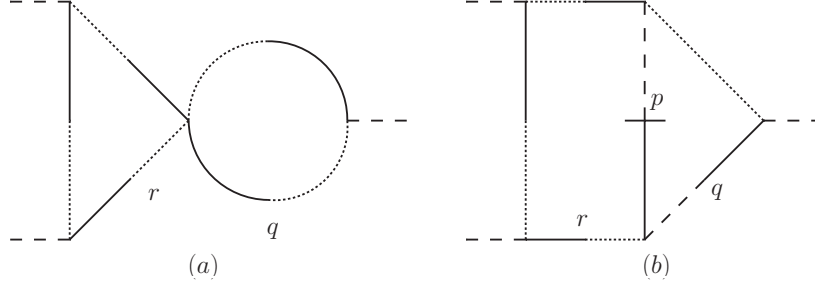


Figure 3: (a) A particular class of CL diagrams from Fig.1(a), with only one external gluon on the right. (b) An incomplete pseudo-Feynman diagram which gives something equivalent to (a). The cross bar on the p -line denotes the incomplete numerator.

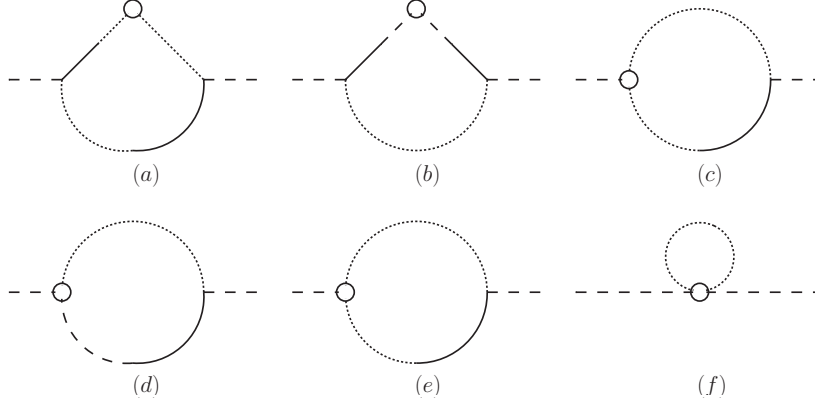


Figure 4: $O(g^4)$ graphs containing $O(g^2)$ counter-term insertions, which are represented by small circles.

The divergent part of 1-loop effective action is known, for example in [17]:

$$\begin{aligned}
 & -C_G \frac{g^2}{16\pi^2} \frac{1}{\epsilon} \left[-\frac{11}{12} F_{ij}^{a2} - \frac{1}{6} F_{0i}^{a2} \right. \\
 & \left. -\frac{4}{3} (F_{0i}^a \{\partial_j A_i^a - g f^{abc} A_i^b A_j^c\} + (1/2) E_i^{a2} + E_i^a \partial_i A_0^a + (u_i^a + \partial_i c^{*a}) \partial_i c^a) \right]
 \end{aligned}
 \tag{4.1}$$

where $F^a_{\mu\nu}$ is defined in (A2), C_G is the Casimir of the colour group, c is the ghost field, and the source u_i^a is introduced in (A1). The Feynman graphs and pseudo-Feynman graphs together have to have the right UV divergences to be cancelled by these counter-terms. In Fig.4 we show some $O(g^4)$ graphs with counter-term insertions deduced from (4.1). For our present purposes, the only relevant term in (4.1) is the second one.

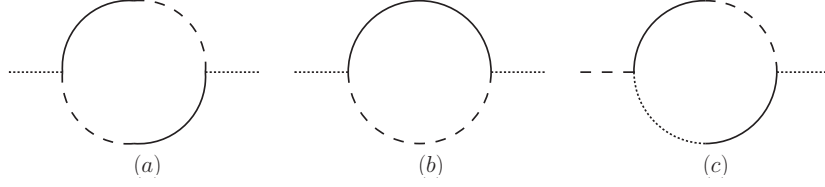


Figure 5: 1-loop self-energy graphs. (a) and (b) are Coulomb line self-energy graphs.

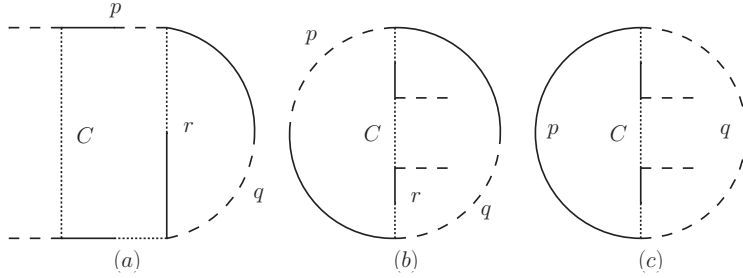


Figure 6: $O(g^4)$ graphs containing self-energy subgraphs. (c) is an ordinary, unambiguous, Feynman graph. (b) is a pseudo-Feynman graph of the type of Fig.2(a), and Fig.6(a) is of the type of Fig.3(b). According to the prescription, in Fig.6(a) the integrals are to be done as in equation (1.5), giving zero; but in Fig.6(b) the integrals are to be done as in equation (1.3), that is integrating over the self-energy sub-graph first.

We begin with Coulomb self-energy sub-graphs. The 1-loop graphs which give the counter-terms are shown in Fig.5(a) and (b). There is an important cancellation between these two graphs, which ensures that their sum is proportional to \mathbf{K}^2 , which is necessary in order to cancel one of the two denominators from Coulomb propagators. Fig.6(c) is a 2-loop Feynman graph which contains Fig.5(b) as a sub-graph. Fig.6(b) is a pseudo-Feynman graph (equivalent to a contribution from the V_2 CL term) which contains Fig.5(a) as a sub-graph. Graph 6(b) comes with the prescription that the p_0 -integral is to be done first, then the r_0 -integral; and this is just what is required for renormalization, where divergent sub-integrations are to be done first.

Next we turn to UV divergent triangle sub-graphs. Fig.7 shows two such graphs. Fig.7(b) is an ordinary, unambiguous Feynman graph, in which p , p' and q lines make up a UV divergent triangle sub-graph. Fig.7(a) is a pseudo-Feynman graph, of the class of Fig.2(a), also with a UV divergent sub-graph triangle. In Fig.7(a), the p_0 -integral is to be done before the r_0 -integral, and this is the order needed for the cancellation of the UV divergence (by the counter-term in Fig.4(e)).

Fig.8 shows three more graphs, in which the triangle formed by the p, q, q'

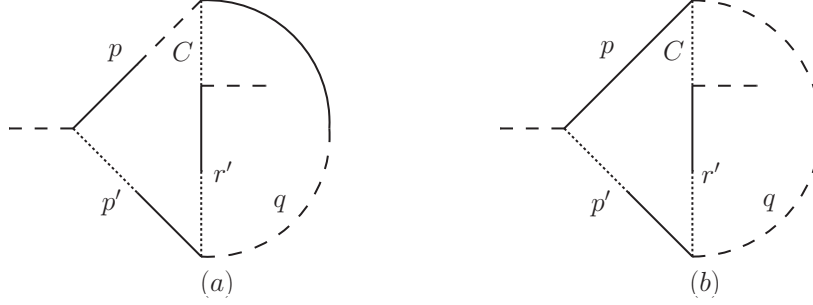


Figure 7: $O(g^4)$ graphs containing UV divergent triangle sub-graphs, made up of the lines p, p', q . (b) is an ordinary, unambiguous Feynman graph. (a) is a pseudo-Feynman graph, of the class in Fig.2(a), derived from a CL term. In (a), the p_0 -integral is to be done before the r_0 one.

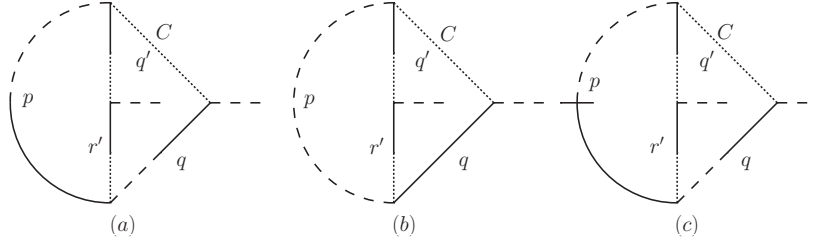


Figure 8: $O(g^4)$ graphs containing UV divergent triangle sub-graphs made up of the lines p, q, q' . (b) is an ordinary, unambiguous Feynman graph. (a) is of the class in Fig.2(b) and so is zero. (c) is derived from W_1 in (2.11), and cancels the UV divergence in (b). In each graph, the symbol C draws attention to the position of the G_0 propagator in (2.3). The bar on the p -propagator in (c) indicated that only the δ_{ij} term in the numerator is retained.

lines is an UV divergent sub-graph. Again Fig.8(b) is an ordinary unambiguous Feynman graph. Fig.8(a) is of the class in Fig.2(b) and so gets zero contribution from W_2 in (2.11). But there is a relevant graph Fig.8(c) of the class of Fig.3(b), coming from W_1 in (2.11). The difference between (a) and (c) lies in a factor $P_i P_j / \mathbf{P}^2$ and because of the transversality of the q propagator this can be replaced by $P_i R'_j / \mathbf{P}^2$, and this contributes only an UV finite part to the sub-integration. Thus the UV divergent sub-integrations in (b) and (c) do combine as they should, and in fact cancel each other; so there is no need for a counter-term.

We have verified (at $O(g^4)$) the existence of sub-graphs with UV divergences necessary to cancel the counter-terms in Fig.4(a), (b), (c) and (e). In the cases of (d) and (f), no ambiguous graphs are involved.

5 Energy divergences

Another unfortunate feature of the Coulomb gauge is the presence, in individual graphs, of energy-divergences, that is integrals of the form

$$\int dr_0 F(r_0, \mathbf{R}) \quad (5.1)$$

where

$$F \rightarrow F(\mathbf{R}) \quad \text{as} \quad r_0 \rightarrow \infty. \quad (5.2)$$

Some, but not all, of these energy divergences occur in just the same graphs which have the ambiguous integrals described in section 1, that is (to 2-loop order) graphs containing only two transverse gluon lines. The ambiguous integrals show up when doing the integrals in the order

$$\int d^3\mathbf{R} d^3\mathbf{P} dr_0 dp_0. \quad (5.2)$$

The energy divergences appear from the order

$$\int d^3\mathbf{R} dr_0 d^3\mathbf{P} dp_0 \quad (5.3)$$

which is the order required by renormalization theory.

It is expected that these energy divergences should cancel when sets of graphs are combined. This has been confirmed at $O(g^4)$ in [17] but only when the sub-graphs are quark loops. Then, the $O(g^2)$ effective action is gauge-invariant, obeying Ward identities. The analysis in [17] made use of these Ward identities. We want to extend the argument to cover gluon loop sub-graphs. Then the $O(g^2)$ effective action is in general not gauge-invariant, but only BRST invariant, so it is not obvious that the cancellation of energy divergences is as simple. The divergent parts are shown in equation (4.1), and indeed they are not all gauge invariant. However, the only one relevant to the energy divergences is the second term in (4.1), which is gauge invariant.

The counter-term graphs in Fig.4 exhibit the energy divergences clearly, and the 2-loop graphs whose sub-divergences Fig.4 cancel are the ones with energy divergences.

In order to extend the argument of [17] to gluon loop sub-graphs, we need to show that these sub-graphs at high energy obey Ward identities, not just BRST identities. The divergent parts obey Ward identities because the ghost graphs shown in Fig.9 are UV convergent. In order to extend the argument to high-energy limits, we need to verify that the graphs like those Fig.9 are also suppressed at high energy. The graphs in Fig.9 have open ghost lines terminating at the ghost field c^a and the sources u_0^a, u_i^a or v_i^a , which appear in the original Lagrangian in equation (A1) in Appendix.

The graphs in Fig.9 all appear by power counting to be linearly divergent in the UV, but they are each proportional to P_i (because the gluon line k is transverse), and so are at most logarithmically divergent. But in (b) there are

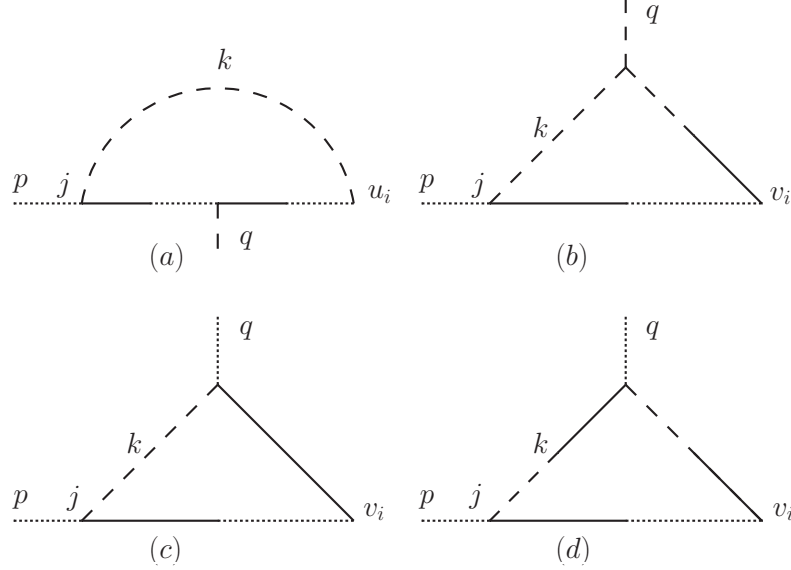


Figure 9: Graphs contributing to the BRST identities. These graphs contain open ghost lines, terminating at the Zinn-Justin source u_0^a, u_i^a or v_i^a , and the bottom line is a ghost line. The integrals appear by power counting to be linearly divergent in the UV, but they all develop external spatial momenta in their numerators which cause them to be UV convergent. (b), (c) and (d) are energy-suppressed but the example in (a) is not.

three spatial indices, i, j, k and so there must be another external spatial momentum. And there is further cancellation between (c) and (d), which produces an extra power of external spatial momenta. Thus (b) and (c)+(d) are actually UV convergent (as indicated by the absence of trilinear ghost counter-terms in (4.1)). Also, to balance dimensions, we expect at least one power of p_0 in the denominator, suppressing the integrals at high-energy (that is $p_0 \gg |\mathbf{P}|, |\mathbf{Q}|, q_0$). In Appendix B, we show this suppression in more detail.

The above suppression does not occur in all ghost graphs. Fig.9(a) is UV convergent, but proportional only to one power of external spatial momenta (Q_i), and is in fact independent of p_0 and q_0 . But, unlike (b), (c) and (d), graphs like (a) with u_i^a or u_0^a sources do not enter into the BRST identities multiplied by a factor of energy, and so are irrelevant to the high-energy behaviour of the vertex parts.

6 Conclusion

To two loop order, the existence of the Christ-Lee terms in the Hamiltonian for Coulomb gauge QCD seems to be a problem for renormalization, because

they do not have sub-graph structure of ordinary Feynman graphs. We have shown that this difficulty is overcome, by re-expressing the Christ-Lee terms as pseudo-Feynman graphs. We have also shown how energy divergences cancel between Feynman and pseudo-Feynman graphs.

However, to three-loop order there may remain problems about the renormalization of the Christ-Lee terms, when ultra-violet divergent sub-graphs are inserted into the ambiguous 2-loop graphs. This question has been discussed in section 5(ii) of [20] and in [21], [22].

This paper is only about perturbation theory. An important open question is whether there are related complications in, for example, lattice calculations. Certainly, the Hamiltonian must be correctly ordered, and this generates CL terms.

Appendix A - Feynman rules

The Lagrangian density, in phase space form, which we use is

$$-\frac{1}{4}F_{ij}^a F_{ij}^a - \frac{1}{2}E_i^a E_i^a + E_i^a F_{0i}^a - \frac{1}{2\alpha}(\partial_i A_i^a)^2 \\ + (u_i^a + \partial_i c^{a*})D_i^{ab}c^b + u_0^a D_0^{ab}c^b - gf^{abc}v_i^a E_i^b c^c - \frac{1}{2}gf^{abc}K^a c^b c^c \quad (A1)$$

where $\mu = (0, i)$, c is the ghost field, u_i^a, u_0^a, v_i^a and K^a are sources. For $\alpha \neq 0$, the equations of motion are non-singular, and we can derive Feynman rules. Then the limit $\alpha \rightarrow 0$ is to be taken, to get the Feynman rules for the Coulomb gauge. In (A1)

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \quad (A2)$$

and

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - gf^{abc}A_\mu^c \quad (A3)$$

In (A1), A_0^a appears linearly and with no time derivative, so it may be integrated out; although we state Feynman rules in the form in which A_0^a is retained. If it is integrated out, it enforces the Coulomb law

$$D_i^{ab}E_i^b = 0. \quad (A4)$$

From (2.1), this gives

$$E_i^a(\mathbf{X}) = E_i^{Ta}(\mathbf{X}) + \int d^3\mathbf{Y} G_i^{ab}(\mathbf{X}, \mathbf{Y}; \mathbf{A}) \rho^b(\mathbf{Y}) \quad (A5)$$

where $\partial_i E_i^{Ta} = 0$ and

$$\rho^a = gf^{abc}A_i^b E_i^{Tc}. \quad (A6)$$

Using these equations and (2.4),

$$\frac{1}{2} \int d^3\mathbf{X} E_i^a E_i^a = \frac{1}{2} \int d^2\mathbf{X} E_i^{Ta} E_i^{Ta} + \frac{1}{2} \int d^3\mathbf{X} d^3\mathbf{X}' \rho^a(\mathbf{X}) C^{ab}(\mathbf{X}, \mathbf{X}'; \mathbf{A}) \rho^b(\mathbf{X}') \quad (A7)$$









i  j	$-\frac{1}{k^2 + i\eta} \left(\delta_{ij} - \frac{K_i K_j}{K^2} \right)$
0  0	$-\frac{1}{K^2}$
E_m  A_j	$-\frac{i k_0}{k^2 + i\eta} \left(\delta_{mj} - \frac{K_m K_j}{K^2} \right)$
A_i  E_m	$\frac{i k_0}{k^2 + i\eta} \left(\delta_{im} - \frac{K_i K_m}{K^2} \right)$
m  n	$-\frac{K^2}{k^2 + i\eta} \left(\delta_{mn} - \frac{K_m K_n}{K^2} \right)$
E_m  A_0	$\frac{i K_m}{K^2}$
A_0  E_n	$-\frac{i K_n}{K^2}$
	$\delta_{ij} - \frac{K_i K_j}{K^2}$ <i>the transverse projection operator</i>

Figure 10: The Feynman rules for propagators. The general rule is that dashed lines represent A_i^a , dotted lines represent A_0^a and continuous lines represent E_i^a .

The second term in (A7) is the non-abelian Coulomb potential operator. How to order the operators (which are all at the same time) in (A7) is one of the questions which lead to the CL terms V_1 and V_2 .

The Feynman rules for (A1) are derived for instance in [17]. They are shown in Fig.10 and Fig.11. In the Coulomb gauge, ghost lines are identical with parts of Coulomb lines. In fact closed ghost loops just cancel closed Coulomb loops. Therefore, only open ghost lines are required, where the ghost is attached to one of the sources $u_i^a, u_0^a, \eta_i^a, K^a$ which appear in (A1), and are shown in Fig.11.

Appendix B - High energy limits

Here we examine the high energy limit of ghost Feynman graphs in more detail than in Section 5. We take as an example Fig.9 (c) and (d). The sum of these two graphs contains the integral

$$P_j \int d^3\mathbf{K} dk_0 \frac{T_{jl}(\mathbf{K})}{k_0^2 - \mathbf{K}^2} \frac{T_{li}(\mathbf{K}')}{k_0'^2 - (\mathbf{K}')^2} \frac{1}{(\mathbf{K} - \mathbf{P})^2} \{\mathbf{K}'^2 - k_0 k_0'\}. \quad (B1)$$

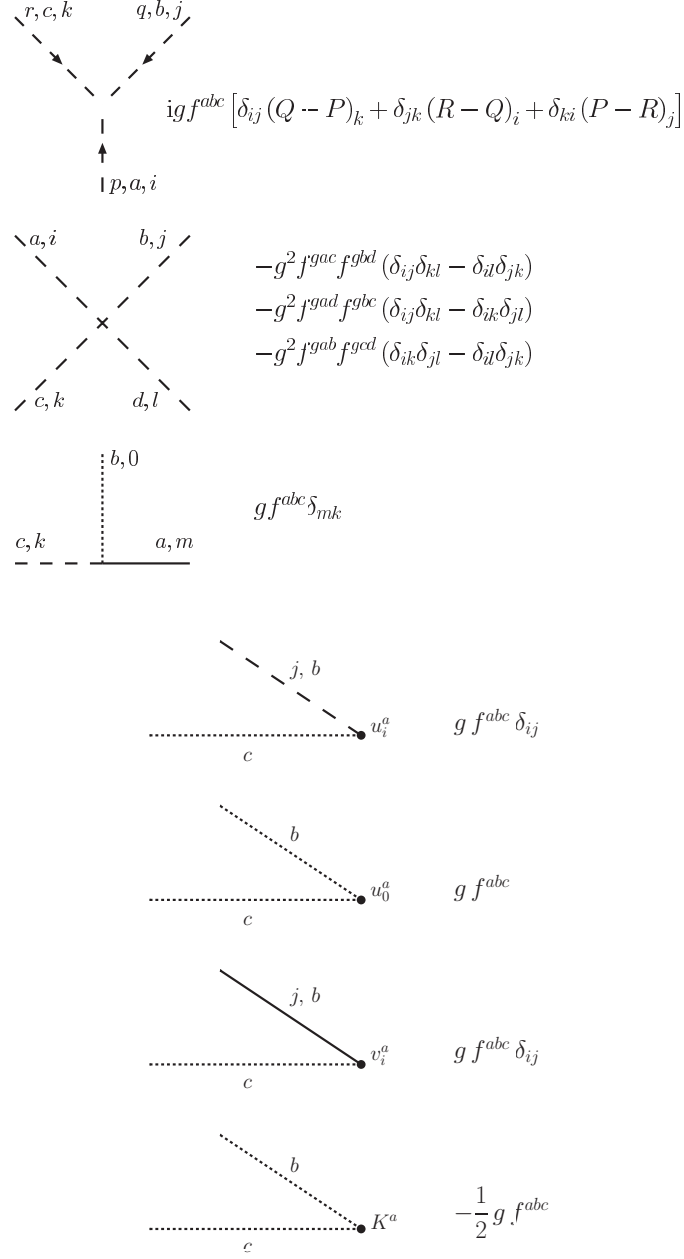


Figure 11: Feynman rules for vertices. The last four involve open ghost lines (with colour index c) and ghost sources.

Doing the k_0 integral gives

$$i\pi P_j \int d^3\mathbf{K} \frac{1}{q_0 - (|\mathbf{K}| + |\mathbf{K}'|)} \frac{1}{(\mathbf{K} - \mathbf{P})^2} \left(\frac{|\mathbf{K}'| - |\mathbf{K}|}{|\mathbf{K}|} \right) T_{jl}(\mathbf{K}) T_{li}(\mathbf{K}') \\ + (q_0 \rightarrow -q_0). \quad (B2)$$

We are concerned with the limit $q_0 \gg |\mathbf{P}|, |\mathbf{Q}|$ and this comes from the region of integration $|\mathbf{K}| \gg |\mathbf{P}|, |\mathbf{Q}|$. So the high-energy limit of (B2) is

$$\frac{i\pi P_j}{q_0} \int d^3\mathbf{K} \frac{1}{(\mathbf{K} - \mathbf{P})^2} \left(\frac{|\mathbf{K}'| - |\mathbf{K}|}{|\mathbf{K}|} \right) T_{jl}(\mathbf{K}) T_{li}(\mathbf{K}') + (q_0 \rightarrow -q_0) + o(1/q_0). \quad (B3)$$

provided that this \mathbf{K} integral is convergent. It is convergent because the factor $(|\mathbf{K}'| - |\mathbf{K}|)$ provides at least one power of \mathbf{Q} and by rotational invariance there must be another factor of \mathbf{P} or \mathbf{Q} . So (B1) tends to zero faster than $1/q_0$ at high energy.

By a similar argument, the graph in Fig.9(b) is shown to have a high-energy limit of the form

$$(1/q_0) Q_j M_{jli}(\mathbf{P}, \mathbf{Q}) \quad (B4)$$

where M is a third rank tensor with dimensions of a momentum..

We are grateful to Dr. G. Duplančić for e-drawing the figures.

Bibliography

1. D Zwanziger, Nucl. Phys. **B518** 237 (1998)
2. A Cucchieri and D Zwanziger, Phys. Rev. **D56** 014002 (2002)
3. A P Szczepaniak and E S Swanton, Phys. Rev. **D65** 025012 (2002)
4. S Sziegel, G Krein and R S Marques de Carvalho, Brazilian J. Of Phys **34** No.1 (2004)
5. R Alkofer, M Kloker, A Krassnig and F Wagenburn, Phys. Tev. Lett. **96** 022001 (2006)
6. A Nakamura and T Saito, Prog. Theor. Phys. **15** 189 (2005)
7. H Reinhard and C Feuchter, Phys. Rev. **D71** 105002 (2005)
8. K Langfeld and L Moyaerts, Phys. Rev. **D70** 074507 (2004)
9. D Epple, H Reinhardt and W Schleifenbaum, Phys. Rev. **D75** 045011 (2007)

10. G Burgio, M Quandt and H Reinhardt, Phys. Rev. Lett. **102** 032002 (2009)
11. D R Campagnari, H Reinhardt and A Weber, Phys. Rev. **D80** 025005 (2009)
12. H Reinhardt, D R Campagnari, D Epple, M Leder, M Pak and W Schleifenbaum, arXiv 0807.4635
13. Y Nakagawa, A Voigt, E-M Ilgenfritz, M Müller-Preussker, A Nakamura, T Saito, A Sternbeck and H Toki, arXiv:0902.4321 [hep-lat] (2009)
14. R N Mohapatra, Phys. Rev **D4** 378 and 1007 (1971)
15. N H Christ and T D Lee, Phys. Rev. **D22** 939 (1980)
16. A Andraši and J C Taylor Eur. Phys. J C **41** 377 (2005)
17. A Andraši and J C Taylor, Annals of Physics **324** 2179 (2009)
18. H Cheng and E-C Tsai, Chinese Journal of Physics **25** 95 (1987)
19. H Cheng and E-C Tsai, Phys. Lett **B176** (1986) 130, Phys. Rev. Lett. **57** 511 (1986)
20. Paul Doust, Annals of Physics **177** 169 (1987)
21. P J Doust and J C Taylor, Physics Letters **197** 232 (1987)
22. A Andraši and J C Taylor, Annals of Physics **326** 1053 (2011)